Transitions among States behind Interactive Agent Model

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Abstract

In this paper, we introduce a simple interactive agent mechanism, where the distribution of returns generated from the mechanism match stylized facts in financial markets. We introduce one more key factor, the length of time horizon on performance evaluations between strategies, which also has a significant influence on price fluctuations. To investigate the transitions among states, we introduce a Markov transition matrix, Perron-Frobenius transition matrix, and Inertia. Our simulation results show the stickiness of states switching from one to another, and the longer length of time horizon on performance evaluations would generate more complex dynamic price fluctuations. We link our simple heterogeneous agent mechanism with Markov trajectory entropy and provide a total score and probability density functions of representations under two states as applications for the mechanism.

Key terms: interactive agent mechanism; Perron-Frobenius transition matrix; Inertia

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1 Introduction

André Kostolany, German stock market guru, wrote in his reminiscence *Die Kunst, über Geld nachzudenken* discussing the phase transitions of financial markets at the section *Das Ei des Kostolany*. He suggested investors buy when stocks are extremely underpriced, hold them during upward trend, and leave market when their prices are exaggerated.

Since the turn of the century, several researchers have introduced and developed heterogeneous interacting agent models and have estimated the parameters of these models. However, there are only a few studies that have investigated the transition among states in heterogeneous interacting agent model.[1] Cheng and Kim (2017) introduced two key factors which would affect price fluctuations: (1) the risk tolerance of fundamentalists and (2) the relative funding rate of positive-feedback traders versus fundamentalists, which would affect price fluctuations. In this paper, one more factor is introduced, length of time horizon on performance evaluations between strategies, to interpret intensity of choice to switch strategies proposed as Brock and Hommes (1997).

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[1] Most studies focus on generating phenomena mimic the stylized face in the financial market. See, for example, Kirman (1993), Lux (1995, 1997, 1998), Lux and Marchesi (1999, 2000), Farmer (2002), and Farmer and Joshi (2002).

The traders' experiences is expected to be related to the length of time horizon on performance evaluations between strategies, where the longer length of time horizon on performance evaluations would generate more complex dynamic price fluctuations. Our simulation results are in line with Greenwood and Nagel (2009), where the shorter l length of time horizon on performance evaluations would generate phenomenon that the state would switch between bubble and crash.

To investigate how the length of time horizon on performance evaluations would affect the transition among states, we include heterogeneous agent models, the noise trader approach, and the leverage cycle in our mechanism, and introduce a Markov transition matrix, Perron-Frobenius transition matrix, and Inertia to investigate how the length of time horizon on performance evaluations would affect the transition among states. We show that the stickiness of states to switch from one to another, and the longer length of time horizon on performance evaluations would generate more complex dynamic price fluctuations. We also connect our mechanism with Markov trajectory entropy proposed by Ekroot and Cover (1993) and provide total score and probability density functions of representations under two states as applications for the mechanism. We propose a problem of connecting multiple systems into one and investigate the state transition behind the system at the last section.

2 Mechanism

We adapt the model (Case 1) from Cheng and Kim (2017). Suppose there is a single stock market that is populated with three types of traders: fundamentalists, positive-feedback traders, and noise traders. All traders in the market are short-sighted and possess beliefs on next period's price for the stock.

Fundamentalists believe stock prices will move back to its fundamental values. They form their expected price based on the differences between fundamental value and current market price, $p_{t+1}^{f} - p_t$, and adjust their expected price each period. The fundamentalists' demand for stock x_t^f and adaptive process p_{t+1}^{f} are shown in equations (1) and (2).

$$x_t^f = \frac{1}{r^f p_t}\left\{\exp\left[\frac{\alpha^f\left(p_{t+1}^{f} - p_t\right)}{r^f p_t} - 1\right] - 1\right\} \tag{1}$$

$$p_{t+1}^{f} = p_t + \nu\left(p_t^* - p_t\right), \text{ where } p_t^* = p^* + \sum_{t=1}^{t} \varepsilon_{2,t} \tag{2}$$

where α^f is the parameter indicating the eagerness of fundamentalists towards profits, r^f is the funding rate fundamentalists face for financing their positions, p^* denotes the initial fundamental value of the stock, ν

captures the speed at which fundamentalists expect the market price to move back to fundamental value, and $\varepsilon_{2,t}$ represents the shock terms resulting from changes of policies and rare events.

Positive-feedback traders chase market trends. They form their expected price based on the differences between their previous expected price and current market price, $p_{t+1}^c - p_t$, and adjust their expected price each period. The positive-feedback traders' demand for stock x_t^c and adaptive process p_{t+1}^c are shown in equations (3) and (4).

$$x_t^c = \frac{1}{r^c p_t}\left\{\exp\left[\frac{\alpha^c\left(p_{t+1}^c - p_t\right) + \beta p_{t+1}^c}{r^c p_t} - 1\right] - 1\right\} \qquad (3)$$

$$p_{t+1}^c = p_t^c + \mu\left(p_t - p_t^c\right) \qquad (4)$$

where α^c is the parameter showing positive-feedback traders' eagerness towards profits, β is the coefficient of the wealth effect, and r^c is the funding rate positive-feedback traders pay for financing their positions, μ is the error correction coefficient representing how sensitive positive-feedback traders correct their expected stock price for the next period.

Accumulated noise traders' demand for stock x_t^n is as follows:

$$x_t^n = \gamma\varepsilon_t, \text{ where } \varepsilon_t = \varepsilon_{1,t} + \varepsilon_{2,t} \qquad (5)$$

$$\varepsilon_{1,t}, \varepsilon_{2,t}\ i.i.d.;\ E\left(\varepsilon_{1,t}\right)=0;\ E\left(\varepsilon_{2,t}\right)=0$$
$$E\left(\varepsilon_{t}\right)=E\left(\varepsilon_{1,t}\right)+E\left(\varepsilon_{2,t}\right)=0 \tag{6}$$

where γ is the reaction strength of noise traders to noisy information and $\varepsilon_{1,t}$ comes from the demand for the stock from liquidity traders and traders with biased belief or sentiments. $\varepsilon_{1,t}$ is normally distributed, and $\varepsilon_{2,t}$ varies with different probability distributions.

There is a market maker whose role is collecting orders, announcing execution prices, and executing transactions. The market price each period is determined by the demand of the stock:

$$p_{t+1}-p_{t}=\theta n\left[\left(1-\kappa_{t}-\xi\right)x_{t}^{f}+\kappa_{t}x_{t}^{c}+\xi x_{t}^{n}\right] \tag{7}$$

where θ is the market sensitivity corresponding to changes in the demand for stocks, n is the total number of traders in the market, and κ_t reflects the population fraction of positive-feedback traders in the market at each period. Noise traders exist in the market with a fixed weight ξ, and the proportion of fundamentalists in the market equals $1-\kappa_t-\xi$.

Different from Cheng and Kim (2017), we let fundamentalists and positive-feedback traders switch to each other's strategy according to the performance differentials of recent history record back to past period $t-l+1$ if $t>l$. The population fractions of fundamentalists and positive-feedback traders are updated each period as follows:

$$\kappa_{t+1} = \frac{1-\xi}{1+\exp\left(\varphi\left(\pi_t^f - \pi_t^c\right)\right)}; \begin{cases} \pi_t^f = \sum_{t=1}^{t} x_t^f \left(p_{t+1} - p_t\right); \pi_t^c = \sum_{t=1}^{t} x_t^c \left(p_{t+1} - p_t\right),\ t \le l \\ \pi_t^f = \sum_{t=t-l+1}^{t} x_t^f \left(p_{t+1} - p_t\right); \pi_t^c = \sum_{t=t-l+1}^{t} x_t^c \left(p_{t+1} - p_t\right),\ t > l \end{cases} \tag{8}$$

where φ is the intensity of choice to switch strategies, π_t^f and π_t^c are the cumulative profits realized by fundamentalists and positive-feedback traders during the length l of history record respectively at each period.

We relax α^f and r^c being able to vary across periods in the mechanism. Fundamentalists become trading more aggressively when they observe more traders switching to positive-feedback traders' strategy, but they trade less aggressively as they are close to dominate the market.

$$\alpha_{t+1}^f = \begin{cases} \alpha_t^f + 0.1,\ \kappa_t > 1-\xi-0.1 \\ \max\left(0.1, \alpha_t^f - 0.1\right),\ \kappa_t < 0.1 \\ \alpha_t^f,\ \text{else} \end{cases} \tag{9}$$

We can see in equation (9) that α^f increases as κ_t is larger than $1-\xi-0.1$, but α^f decreases when κ_t is less than 0.1.[2] r^c also represents for relative funding difficulty of positive-feedback traders versus fundamentalists. As the stock price increases, positive-feedback traders face higher funding rate for financing their positions.

$$r_{t+1}^c = \begin{cases} \min\left[r_t^c + \frac{\left(p_{t+1} - p_t\right)}{1000}, 0.95\right],\ r_{t+1}^c \ge 0.1 \\ 0.1,\ r_{t+1}^c < 0.1 \end{cases} \tag{10}$$

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[2] Here 0.1 is the threshold ϑ to induce fundamentalists having reaction responding to the popularity of their strategy.

We set the minimum and the maximum of r^c equal 0.1 and 0.95 respectively.

3 Results

Similar to Cheng and Kim (2017), we fix all the parameters at the initial settings and allow the length of evaluations on performances l, the initial α^f, and the initial r^c to vary within the considered range with different shock terms based on the normal, student t (t), and α-stable distributions, or GARCH processes. Table 1 shows the initial settings.

3.1 Simulations

The mechanism enable us to generate different scenario in a trial and also remain the characteristics of the model developed in Cheng and Kim (2017). Figure 1 presents the examples of simulations with different initial settings. Figures 2 and 3 show the autocorrelation of (absolute) simulated returns. In the mechanism, simulated returns are weakly autocorrelated while the absolute simulated returns have significant positive and slow decaying autocorrelations. We can see in Table 2 that the simulated returns are most stationary, but reject the null hypothesis of

Kwiatkowski-Phillips-Schmidt-Shin test that the simulated returns are not trend stationary when l equals to 500.

3.2 Transitions among states

Fundamentalists have more chance to dominate the market when they are more willing than positive-feedback traders to take risks. Also, more crises occur as positive-feedback traders have higher funding costs compared to fundamentalists (Cheng and Kim, 2017). We then define different states based on the magnitudes of α_t^f and r_t^c.

$$S_P := \left\{\alpha_t^f,\ r_t^c \middle| \alpha_t^f < 8,\ r_t^c < 0.2\right\} \tag{11}$$

$$S_F := \left\{\alpha_t^f,\ r_t^c \middle| \alpha_t^f \geq 8,\ r_t^c < 0.2\right\} \tag{12}$$

$$S_C := \left\{\alpha_t^f,\ r_t^c \middle| r_t^c \geq 0.2\right\} \tag{13}$$

Equation (11), (12), and (13) present the conditions for states S_P, S_F, and S_C, where positive-feedback traders (resp. fundamentalists) are more able to dominate the market when α_t^f is less (resp. larger) than 8 and more crises occur when r_t^c is larger than 0.2. Figures 4 and 5 present the examples of trajectories of α_t^f and r_t^c with the transitions of states across periods for l equals to 10 and 500, where 1, 2, and 3 represent for S_P, S_C, and S_F respectively. Our results indicate that the larger l would induce more complicate dynamics of state transitions.

3.3 Representations of states transitions

We calculate the Markov transition matrix of states M, and define the Perron-Frobenius transition matrix P with $P_{ij} = 1$ if $M_{ij} > 0$ and $P_{ij} = 1$ if $M_{ij} = 0$ for $i, j = P, F, C$. Figures 6 and 7 are the examples of Markov transition matrix and Perron-Frobenius transition matrix, where our results show the stickiness of states to switch from one to another.

Our mechanism has the property that if a state S_i, where $i = P, F, C$, is reached, the probability that next period is still at the same state S_i is positive. That is, $\Pr\left(S_{i,t+1} \middle| S_{i,t}\right) > 0$ for $i = P, F, C$. Because of this characteristic, we can reduce the number of representations for states transitions to forty-four, which we show in the Appendix 1.

When l is small, market tends to switch between two states (representations before the 13rd). However, the state transitions become more complicated as l goes larger (see Figure 8). Shock terms based on α -stable distributions, or GARCH processes (except FIGARCH processes) would also induce complicate state transitions when l is small (see Figures 9 and 10). In addition, market most likely stay at S_C (the 2nd

representation) when the initial r^c is larger than 0.3 (see Figure 11).

To understand more about the phenomenon, let $T_{l,\vartheta}$ be the set of representations occurred in simulations. If we look at the strategy switch function $\kappa_{t+1} = \frac{1-\xi}{1+\exp\left(\varphi\left(\pi_t^f - \pi_t^c\right)\right)}$, we can see that the population fractions of fundamentalists and positive-feedback traders each period are determined by the cumulative profits realized of fundamentalists and positive-feedback traders during the length l of history record. If l is small, $\pi_t^f - \pi_t^c$ is within a small range such that κ_{t+1} tends not to meet the levels ($\kappa_{t+1} < \vartheta$ or $\kappa_{t+1} > 1-\xi-\vartheta$) which induce fundamentalists to trade less (or more) aggressively. Thus, the state tends to stay at the same state or switches between another states. If l is large, κ_{t+1} has more chances to hit the levels which induce fundamentalists' reactions and we would observe more complicated state transitions. In addition, we would also observe more complicated state transitions if the threshold ϑ is larger (see Figure 12). We conclude that the cardinality of $T_{l,\vartheta}$, $\left|T_{l,\vartheta}\right|$, is smaller (larger) if l is smaller (larger) or ϑ is smaller (larger), $\left|T_{l,\vartheta}\right| \le \left|T_{l',\vartheta'}\right|$ if $l < l'$ or $\vartheta < \vartheta'$.

3.4 Inertia of states

We define the distance between current state S_t and the state in next time periods S_{t+s} as follows.

$$d_{i,t+s}\left(S_t,\ S_{t+s}\right)=\begin{cases}0,\ S_t=S_{t+s}\\1,\ S_t\neq S_{t+s}\end{cases},\ \forall s\in\mathbb{N},\ i=P,C,F \tag{14}$$

Note that if $d_{i,t+s}\left(S_t,\ S_{t+s}\right)$ equals 1 does not indicate that $d_{i,t+s'}\left(S_t,\ S_{t+s'}\right)$ is also equal to 1 for $s'<s$. We then define the inertia $I_{i,t+s}$ for the state starting with S_i, where $i=P,C,F$, in the following:

$$I_{i,t+s}=\sum_{S_{t+s}=S_P,S_C,S_F} d_{i,t+s}\left(S_t,\ S_{t+s}\right)\cdot\Pr\left(S_{t+s}\middle|S_t=S_i\right),\ \forall s\in\mathbb{N} \tag{15}$$

Because of the stickiness of states to switch from one to another, the inertia $I_{i,t+s}$ tells us the probability of escaping the initial state S_t at time period $t+s$. If $I_{i,t+s}$ is larger, the probability that market stays at the same state as S_t is smaller.

If the initial r^c is less than 0.3, the measures of inertia differ with shock terms based on different distributions and processes. For the shock terms based on normal, *t*, and FIGARCH processes, I_C tends to be larger compared to I_P and I_F in the short term, but tends to be smaller in the long term (Figure 13). We also observe the situation that I_P or I_F is the smallest across time period (Figure 14). For the shock terms based on α -stable distributions and GARCH processes, I_C tends to be the smallest across time period (Figure 15). When the initial r^c is larger than 0.3, market most likely stays at S_C. The average inertia of S_C is extremely small and the average inertia of S_P and S_F are close to 1 as time period goes longer (Figure 16).

4 Applications

We introduce total score, entropy of Markov trajectories, and probability density function of representations for two states in this section.

4.1 Total score

We define total score $Total_{t+s}$ as follows,

$$Total_{t+s} = \sum_{i=P,C,F} \sum_{j=P,C,F} w_{ij} \Pr\left(S_{t+s} \middle| S_t = S_i\right). \tag{16}$$

where w_{ij} is the score assigned to the state switching from S_i to S_j.

Let the score matrix W and the Markov transition matrix M as follows,

$$W = \begin{bmatrix} w_{PP} & w_{PC} & w_{PF} \\ w_{CP} & w_{CC} & w_{CF} \\ w_{FP} & w_{FC} & w_{FF} \end{bmatrix}, M = \begin{bmatrix} p_P & q_P & 1-p_P-q_P \\ p_C & q_C & 1-p_C-q_C \\ p_F & q_F & 1-p_F-q_F \end{bmatrix}. \tag{17}$$

We define preference relations of state transition $S_{ki} \succ S_{kj}$ if $w_{ki} < w_{kj}$ for $i, j, k = P, C, F$; $S_{ki} \sim S_{kj}$ if $w_{ki} = w_{kj}$ for $i, j, k = P, C, F$. Let S_{best} be the state transition with highest score ($\underset{i,j=P,C,F}{Max}\left(w_{ij}\right)$) and S_{worst} be the state transition with lowest score ($\underset{i,j=P,C,F}{\min}\left(w_{ij}\right)$). For the rest state transitions, there exists $\lambda \in (0,1)$ such that $S_{rest} \sim \lambda S_{best} + (1-\lambda) S_{worst}$. From our definition, the total score T is equal to $3\left(\frac{1}{3}K_P + \frac{1}{3}K_C + \frac{1}{3}K_F\right)$, where $K_i = p_i w_{iP} + q_i w_{iC} + (1-p_i-q_i) w_{iF}$, $i = P,C,F$. We can see that $\frac{1}{3}K_P + \frac{1}{3}K_C + \frac{1}{3}K_F$ belongs to the convex set with respect to the preference we assign to each state transition. Similar discussions apply to the transitions among finite n states.

Proposition 4.1

If the Markov transition matrix M is positive, T preserves the preferences assigned to state transitions.

We can calculate the total scores to find the optimal choice for a group of initial settings. Suppose we let $w_{FF} = 5$, $w_{PF} = w_{CF} = 2.5$, $w_{PP} = w_{FP} = 0$, $w_{CP} = 1$, $w_{PC} = w_{FC} = -2.5$, $w_{CC} = -5$. We calculate the total scores if the transition matrices are all positive in observed time periods. The optimal choice is the initial settings with the highest total score. We can see in Figure 17 that l equals to 400 is optimal compared to l equals to 100, 200, 300, and 500 within our choices. Note that $r^c \geq 0.3$ is less desirable because the market most likely remain at S_C.

4.2 Markov trajectory entropy

Ekroot and Cover (1993) derived the general closed form solution of Markov trajectory entropy. For a finite state irreducible Markov transition matrix M, the entropy rate $H(\chi) = -\sum_{i,j} \mu_i M_{ij} \log M_{ij}$ where μ is the stationary distribution of the transition matrix satisfying $\mu_j = \sum_i \mu_i M_{ij}$ for all j. Let $M_{i\cdot}$ denote the i^{th} row of the transition matrix, the entropy of the first step of a trajectory originating in state i, $H(M_{i\cdot})$, is $-\sum_j M_{ij} \log M_{ij}$. Thus, the matrix of first step entropies is

$$H^* = \begin{bmatrix} H(M_{1\cdot}) & H(M_{1\cdot}) & \cdots & H(M_{1\cdot}) \\ H(M_{2\cdot}) & H(M_{2\cdot}) & \cdots & H(M_{2\cdot}) \\ \vdots & \vdots & \vdots & \vdots \\ H(M_{m\cdot}) & H(M_{m\cdot}) & \cdots & H(M_{m\cdot}) \end{bmatrix}. \tag{18}$$

Let H be the matrix of Markov trajectory entropies, the diagonal matrix H_Δ associated with H is as follows,

$$H_\Delta = \begin{bmatrix} H_{11} & 0 & 0 & \cdots & 0 \\ 0 & H_{22} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & H_{mm} \end{bmatrix}. \tag{19}$$

Theorem 4.2.1 (Ekroot and Cover, 1993)

For an irreducible Markov chain, the entropy H_{ii} of the random trajectory from state i back to state i is $\frac{H(\chi)}{\mu_i}$.

Theorem 4.2.2 (Ekroot and Cover, 1993)

If M is the transition matrix of an irreducible finite state Markov chain, then the matrix H of trajectory entropies is $K-\tilde{K}+H_{\Delta}$ where $K=(I-M+A)^{-1}(H^{*}-H_{\Delta})$, $\tilde{K}_{ij}=K_{jj}$ for all i, j, and $A_{ij}=\mu_j$ for all i, j.

The entries in the matrix of trajectory entropies indicate the complexity of trajectories from a state i to itself or to the other states j. A larger trajectory entropy H_{ij} for $i, j = P, C, F$ indicates that the trajectory from state i to states j could be more complicated. Figure 18 presents the example of the matrix of Markov trajectory entropies from our mechanism. We can see that the entries in the diagonal matrix of Markov trajectory entropy are much smaller compared to other entries in the matrix, which shows the stickiness of states to switch from one to another. We can also observe that the trajectory could be more complicated from state S_F to state S_P than the trajectory be from state S_P to S_F. That is, the length of trajectory could be longer from state S_F to state S_P compared to the length of trajectory from state S_P to S_F.

4.3 Probability density functions of representations under two states

Let M be the Markov transition matrix of two states S_A and S_B where $M_{AA} = p$ and $M_{BB} = q$.

$$M = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix} \tag{20}$$

Let h be the history of state transitions, $h = \{(S_0 S_1 S_2 \ldots S_N) | S_t = S_A \text{ or } S_B \text{ and } t = 0,1,2,\ldots,N\}$.

(21)

For a given history of state transitions, we could categorize the history into a specific representation (Perron-Frobenius transition matrix). Without loss of generality, we start with the initial state S_A moving for N steps and investigate the probabilities of each representation occurred.

4.3.1 Representation A $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Representation A occurs when remaining at the initial state S_A for the whole N moves, $h = \{(S_0 S_1 S_2 \ldots S_N) | S_t = S_A \text{ and } t = 0,1,2,\ldots,N\}$. The probability $\Pr\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = p^N$ for $N \geq 1$. (22)

4.3.2 Representation B $\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$

Representation B occurs when the initial state S_A remain at the same state for $N-1$ steps but switch to state S_B at the last step, $h = \{(S_0 S_1 S_2 \ldots S_N) | S_t = S_A \text{ for } t = 0,1,2,\ldots,N-1 \text{ and } S_N = S_B\}$. The probability

$$\Pr\left(\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}\right) = \left\langle \begin{matrix} p^{N-1}(1-p), & N \geq 2 \\ 0 & , N = 1 \end{matrix} \right. \quad (23)$$

4.3.3 Representation C $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$

Representation C occurs when the initial state S_A remain at the same state for $N-2$ steps but switch to state S_B for the $(N-1)^{th}$ step and stay at the last step, $h = \{(S_0 S_1 S_2 \ldots S_N) | S_t = S_A \text{ for } t = 0,1,2,\ldots,N-2 \text{ and } S_t = S_B \text{ for } t = N-1, N\}$.

The probability $\Pr\left(\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\right) = \left\langle \begin{matrix} p^{N-2}(1-p)q, & N \geq 3 \\ 0 & , N = 1,2 \end{matrix} \right.$. (24)

4.3.4 Representation D $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Representation D occurs when state S_A remains at the same state and state S_B switches to state S_A both happen at least once in the whole N steps and $N \geq 3$. To be clear, we discuss the cases separately for $N \geq 3$.

(i) $N = 3$

$h = \{(S_A S_A [S_B S_A]), (S_A [S_B S_A] S_A)\}$ where we would have two components S_A or $[S_B S_A]$ in the history of Representation D. When $N = 3$, we have $2!$ choices of having S_A and $[S_B S_A]$ happened once. The probability $\Pr\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = 2!\,p(1-p)(1-q)$ when $N = 3$.

(ii) $N = 4$

$h = \{(S_A S_A S_A [S_B S_A]), (S_A S_A [S_B S_A] S_A), (S_A [S_B S_A] S_A S_A)\}$. When $N = 4$, we have $\frac{3!}{2!1!}$ choices of having S_A happened twice and $[S_B S_A]$ happened once. The probability $\Pr\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = \frac{3!}{2!1!} p^2 (1-p)(1-q)$ when $N = 4$.

(iii) $N = 5$

If S_A happens three times and $[S_B S_A]$ happens once,

$$h=\{(S_AS_AS_AS_A[S_BS_A]),(S_AS_AS_A[S_BS_A]S_A),(S_AS_A[S_BS_A]S_AS_A),(S_A[S_BS_A]S_AS_AS_A)\}$$

which we have $\frac{4!}{3!1!}$ choices. The probability of having S_A happened three times and $[S_BS_A]$ happened once is as follows,

$$\Pr\left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\middle| S_A \text{ 3 times, } [S_BS_A] \text{ once}\right)=\frac{4!}{3!1!}p^3(1-p)(1-q) \text{ when } N=5.$$

If S_A happens once and $[S_BS_A]$ happens twice, $h=\{(S_AS_A[S_BS_A][S_BS_A]),(S_A[S_BS_A]S_A[S_BS_A]),(S_A[S_BS_A][S_BS_A]S_A)\}$ which we have $\frac{3!}{1!2!}$ choices. The probability of having S_A happened once and $[S_BS_A]$ happened twice is as follows,

$$\Pr\left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\middle| S_A \text{ once, } [S_BS_A] \text{ twice}\right)=\frac{3!}{1!2!}p\left[(1-p)(1-q)\right]^2 \text{ when } N=5.$$

The probability of Representation D occurred when $N=5$ is the sum of the probabilities which S_A happens three times and $[S_BS_A]$ happens once or S_A happens once and $[S_BS_A]$ happens twice.

$$\Pr\left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\right)=\frac{4!}{3!1!}p^3(1-p)(1-q)+\frac{3!}{1!2!}p\left[(1-p)(1-q)\right]^2 \text{ when } N=5.$$

(iiii) $N=6$

If S_A happens four times and $[S_BS_A]$ happens once, we have $\frac{5!}{4!1!}$ choices. The probability of having S_A happened four times and $[S_BS_A]$ happened once is as follows,

$$\Pr\left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} \middle| S_A \text{ 4 times, } [S_B S_A] \text{ once}\right) = \frac{5!}{4!1!} p^4 (1-p)(1-q) \text{ when } N=6.$$

If S_A and $[S_B S_A]$ happens twice, we have $\frac{4!}{2!2!}$ choices. The probability of having S_A and $[S_B S_A]$ happened twice is as follows,

$$\Pr\left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} \middle| S_A \text{ twice, } [S_B S_A] \text{ twice}\right) = \frac{4!}{2!2!} p^2 \left[(1-p)(1-q)\right]^2 \text{ when } N=6.$$

The probability of Representation D occurred when $N=6$ is the sum of the probabilities which S_A happens four times and $[S_B S_A]$ happens once or S_A and $[S_B S_A]$ happens twice.

$$\Pr\left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\right) = \frac{5!}{4!1!} p^4 (1-p)(1-q) + \frac{4!}{2!2!} p^2 \left[(1-p)(1-q)\right]^2 \text{ when } N=6.$$

(v) $N=7$

If S_A happens five times and $[S_B S_A]$ happens once, we have $\frac{6!}{5!1!}$ choices. The probability of having S_A happened five times and $[S_B S_A]$ happened once is as follows,

$$\Pr\left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} \middle| S_A \text{ 5 times, } [S_B S_A] \text{ once}\right) = \frac{6!}{5!1!} p^5 (1-p)(1-q) \text{ when } N=7.$$

If S_A happens three times and $[S_B S_A]$ happens twice, we have $\frac{5!}{3!2!}$ choices. The probability of having S_A happened three times and $[S_B S_A]$ happened twice is as follows,

$$\Pr\left(\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\middle| S_A \text{ 3 times, } [S_B S_A] \text{ twice}\right) = \frac{5!}{3!2!} p^3 \left[(1-p)(1-q)\right]^2 \text{ when } N = 7.$$

If S_A happens once and $[S_B S_A]$ happens three times, we have $\frac{4!}{1!3!}$ choices. The probability of having S_A happened once and $[S_B S_A]$ happened three times is as follows,

$$\Pr\left(\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\middle| S_A \text{ once, } [S_B S_A] \text{ 3 times}\right) = \frac{4!}{1!3!} p \left[(1-p)(1-q)\right]^3 \text{ when } N = 7.$$

The probability of Representation D occurred when $N = 7$ is the sum of the probabilities which S_A happens five times and $[S_B S_A]$ happens once, S_A happens three times and $[S_B S_A]$ happens twice, or S_A happens once and $[S_B S_A]$ happens three times.

$$\Pr\left(\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\right) = \frac{6!}{5!1!} p^5 (1-p)(1-q) + \frac{5!}{3!2!} p^3 \left[(1-p)(1-q)\right]^2 + \frac{4!}{1!3!} p \left[(1-p)(1-q)\right]^3$$

when $N = 7$.

(vi) $N = 8$

If S_A happens six times and $[S_B S_A]$ happens once, we have $\frac{7!}{6!1!}$ choices. The probability of having S_A happened six times and $[S_B S_A]$ happened once is as follows,

$$\Pr\left(\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\middle| S_A \text{ 6 times, } [S_B S_A] \text{ once}\right) = \frac{7!}{6!1!} p^6 (1-p)(1-q) \text{ when } N = 8.$$

If S_A happens four times and $[S_B S_A]$ happens twice, we have $\frac{6!}{4!2!}$ choices. The probability of having S_A happened four times and $[S_B S_A]$ happened twice is as follows,

$$\Pr\left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} \middle| S_A \text{ 4 times, } [S_B S_A] \text{ twice}\right) = \frac{6!}{4!2!} p^4 \left[(1-p)(1-q)\right]^2 \text{ when } N = 8.$$

If S_A happens twice and $[S_B S_A]$ happens three times, we have $\frac{5!}{2!3!}$ choices. The probability of having S_A happened twice and $[S_B S_A]$ happened three times is as follows,

$$\Pr\left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} \middle| S_A \text{ twice, } [S_B S_A] \text{ 3 times}\right) = \frac{5!}{2!3!} p^2 \left[(1-p)(1-q)\right]^3 \text{ when } N = 8.$$

The probability of Representation D occurred when $N = 8$ is the sum of the probabilities which S_A happens six times and $[S_B S_A]$ happens once, S_A happens four times and $[S_B S_A]$ happens twice, or S_A happens twice and $[S_B S_A]$ happens three times.

$$\Pr\left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\right) = \frac{7!}{6!1!} p^6 (1-p)(1-q) + \frac{6!}{4!2!} p^4 \left[(1-p)(1-q)\right]^2 + \frac{5!}{2!3!} p^2 \left[(1-p)(1-q)\right]^3$$

when $N = 8$.

(vii) N steps

From above discussion for Representation D, we can solve the probability of Representation D occurred as follows.

$$\Pr\left(\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\right)=\begin{cases}\sum_{n=N-\frac{N-1}{2}+1}^{N}\frac{(n-1)!}{\left[N-2(N-n+1)\right]!(N-n+1)!}p^{N-2(N-n+1)}\left[(1-p)(1-q)\right]^{N-n+1}, & N\geq 3,\ N\text{ odd}\\ \sum_{n=N-\left(\frac{N}{2}-1\right)+1}^{N}\frac{(n-1)!}{\left[N-2(N-n+1)\right]!(N-n+1)!}p^{N-2(N-n+1)}\left[(1-p)(1-q)\right]^{N-n+1}, & N\geq 3,\ N\text{ even}\\ 0 & ,\ N=1,2\end{cases}$$

$$=\begin{cases}\sum_{n=\frac{N+3}{2}}^{N}\frac{(n-1)!}{(2n-N-2)!(N-n+1)!}p^{2n-N-2}\left[(1-p)(1-q)\right]^{N-n+1}, & N\geq 3,\ N\text{ odd}\\ \sum_{n=\frac{N+4}{2}}^{N}\frac{(n-1)!}{(2n-N-2)!(N-n+1)!}p^{2n-N-2}\left[(1-p)(1-q)\right]^{N-n+1}, & N\geq 3,\ N\text{ even}\\ 0 & ,\ N=1,2\end{cases}$$

$$=\begin{cases}\sum_{n=N-\left\lceil\frac{N-1}{2}\right\rceil+1}^{N}\frac{(n-1)!}{(2n-N-2)!(N-n+1)!}p^{2n-N-2}\left[(1-p)(1-q)\right]^{N-n+1}, & N\geq 3\\ 0 & ,\ N=1,2\end{cases}$$

Thus, the probability

$$\Pr\left(\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\right)=\begin{cases}\sum_{n=N-\left\lceil\frac{N-1}{2}\right\rceil+1}^{N}\frac{(n-1)!}{(2n-N-2)!(N-n+1)!}p^{2n-N-2}\left[(1-p)(1-q)\right]^{N-n+1}, & N\geq 3\\ 0 & ,\ N=1,2\end{cases} \tag{25}$$

4.3.5 Representation E $\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$

Representation E occurs only for $N=1$, $h=\{(S_A S_B)\}$. The probability

$$\Pr\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right)=\left\langle\begin{matrix}0 & ,\ N\geq 2\\ 1-p, & N=1\end{matrix}\right.. \tag{26}$$

4.3.6 Representation F $\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$

Representation F occurs when the initial state S_A switches to state

S_B in the first move and stays at state S_B for the rest moves, $h=\{(S_0S_1S_2\ldots S_N)|S_0=S_A \text{ and } S_t=S_B \text{ for } t=1,2,\ldots,N\}$. The probability

$$\Pr\left(\begin{bmatrix}0&1\\0&1\end{bmatrix}\right)=\left\langle\begin{matrix}(1-p)q^{N-1}, & N\geq 2\\ 0 & , N=1\end{matrix}\right. . \qquad (27)$$

4.3.7 Representation G $\begin{bmatrix}0&1\\1&0\end{bmatrix}$

Representation G occurs when states S_A and state S_B switch to each other without staying at the same state, $h=\{(S_0S_1S_2\ldots S_N)|S_t=S_A \text{ for } t \text{ even and } S_t=S_B \text{ for } t \text{ odd}\}$. The probability

$$\Pr\left(\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)=\left\langle\begin{matrix}(1-p)^{\left\lceil\frac{N+1}{2}\right\rceil}(1-q)^{\left\lfloor\frac{N}{2}\right\rfloor}, & N\geq 2\\ 0 & , N=1\end{matrix}\right. . \qquad (28)$$

4.3.8 Representation H $\begin{bmatrix}0&1\\1&1\end{bmatrix}$

Representation H occurs when the initial state S_A switches to state S_B in the first move with the rest moves similar to Representation D but starting with initial state S_B moving for $N-1$ steps.

Thus, the probability

$$\Pr\left(\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}\right)=\begin{cases}(1-p)\displaystyle\sum_{n=N-\left\lceil\frac{N-2}{2}\right\rceil}^{N-1}\frac{(n-1)!}{(2n-N-1)!(N-n)!}q^{2n-N-1}\left[(1-q)(1-p)\right]^{N-n}, & N\geq 4\\ q(1-p)(1-q) & ,\ N=3\\ 0 & ,\ N=1,2\end{cases}\tag{29}$$

4.3.9 Representation I $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

Representation I is the most complicated case among all the representations, and the probability of Representation I occurred is equal to one minus the sum of probabilities that Representations A, B, C,...,H occur. Thus, the probability

$$\Pr\left(\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right)=\begin{cases}1-p^N-p^{N-1}(1-p)-p^{N-2}(1-p)q-(1-p)q^{N-1}-(1-p)^{\left\lceil\frac{N+1}{2}\right\rceil}(1-q)^{\left\lceil\frac{N}{2}\right\rceil}\\ -\displaystyle\sum_{n=N-\left\lceil\frac{N-1}{2}\right\rceil+1}^{N}\frac{(n-1)!}{(2n-N-2)!(N-n+1)!}p^{2n-N-2}\left[(1-p)(1-q)\right]^{N-n+1}\\ -(1-p)\displaystyle\sum_{n=N-\left\lceil\frac{N-2}{2}\right\rceil}^{N-1}\frac{(n-1)!}{(2n-N-1)!(N-n)!}q^{2n-N-1}\left[(1-q)(1-p)\right]^{N-n}, & N\geq 4\\ 0 & ,\ N=1,2,3\end{cases}\tag{30}$$

Similarly, if we start with initial state S_B, which $M_{\text{initial }S_B}=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}M_{\text{initial }S_A}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, we can apply the formulas discussed above to derive the probability density function of each representation starting with initial state S_B.

Proposition 4.3

Let M be two-state Markov transition matrix, $M = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$.

(1) The probability density function of representations with initial state S_A moving for N steps is as follows.

A. $\Pr\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \middle| S_0 = S_A \right) = p^N,\ N \geq 1$

B. $\Pr\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \middle| S_0 = S_A \right) = \begin{cases} p^{N-1}(1-p), & N \geq 2 \\ 0 & , N = 1 \end{cases}$

C. $\Pr\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \middle| S_0 = S_A \right) = \begin{cases} p^{N-2}(1-p)q, & N \geq 3 \\ 0 & , N = 1,2 \end{cases}$

D.

$$\Pr\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \middle| S_0 = S_A \right) = \begin{cases} \displaystyle\sum_{n=N-\left[\frac{N-1}{2}\right]+1}^{N} \frac{(n-1)!}{(2n-N-2)!(N-n+1)!} p^{2n-N-2} \left[(1-p)(1-q)\right]^{N-n+1}, & N \geq 3 \\ 0 & , N = 1,2 \end{cases}$$

E. $\Pr\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \middle| S_0 = S_A \right) = \begin{cases} 0 & , N \geq 2 \\ 1-p, & N = 1 \end{cases}$

F. $\Pr\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \middle| S_0 = S_A \right) = \begin{cases} (1-p)q^{N-1}, & N \geq 2 \\ 0 & , N = 1 \end{cases}$

G. $\Pr\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \middle| S_0 = S_A \right) = \begin{cases} (1-p)^{\left[\frac{N+1}{2}\right]}(1-q)^{\left[\frac{N}{2}\right]}, & N \geq 2 \\ 0 & , N = 1 \end{cases}$

H.

$$\Pr\left(\begin{bmatrix}0&1\\1&1\end{bmatrix}\middle|S_0=S_A\right)=\begin{cases}(1-p)\displaystyle\sum_{n=N-\left\lceil\frac{N-2}{2}\right\rceil}^{N-1}\frac{(n-1)!}{(2n-N-1)!(N-n)!}q^{2n-N-1}\left[(1-q)(1-p)\right]^{N-n}, & N\geq 4\\ q(1-p)(1-q) & ,\ N=3\\ 0 & ,\ N=1,2\end{cases}$$

I.

$$\Pr\left(\begin{bmatrix}1&1\\1&1\end{bmatrix}\middle|S_0=S_A\right)=\begin{cases}1-p^N-p^{N-1}(1-p)-p^{N-2}(1-p)q-(1-p)q^{N-1}-(1-p)^{\left\lceil\frac{N+1}{2}\right\rceil}(1-q)^{\left\lceil\frac{N}{2}\right\rceil}\\ -\displaystyle\sum_{n=N-\left\lceil\frac{N-1}{2}\right\rceil+1}^{N}\frac{(n-1)!}{(2n-N-2)!(N-n+1)!}p^{2n-N-2}\left[(1-p)(1-q)\right]^{N-n+1}\\ -(1-p)\displaystyle\sum_{n=N-\left\lceil\frac{N-2}{2}\right\rceil}^{N-1}\frac{(n-1)!}{(2n-N-1)!(N-n)!}q^{2n-N-1}\left[(1-q)(1-p)\right]^{N-n}, & N\geq 4\\ 0 & ,\ N=1,2,3\end{cases}$$

(2) The probability density function of representations with initial state S_B moving for N steps is as follows.

A'. $\Pr\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\middle|S_0=S_B\right)=q^N,\ N\geq 1$

B'. $\Pr\left(\begin{bmatrix}0&0\\1&1\end{bmatrix}\middle|S_0=S_B\right)=\begin{cases}q^{N-1}(1-q), & N\geq 2\\ 0 & ,\ N=1\end{cases}$

C'. $\Pr\left(\begin{bmatrix}1&0\\1&1\end{bmatrix}\middle|S_0=S_B\right)=\begin{cases}q^{N-2}(1-q)p, & N\geq 3\\ 0 & ,\ N=1,2\end{cases}$

D'.

$$\Pr\left(\begin{bmatrix}0&1\\1&1\end{bmatrix}\middle|S_0=S_B\right)=\begin{cases}\displaystyle\sum_{n=N-\left\lceil\frac{N-1}{2}\right\rceil+1}^{N}\frac{(n-1)!}{(2n-N-2)!(N-n+1)!}q^{2n-N-2}\left[(1-q)(1-p)\right]^{N-n+1}, & N\geq 3\\ 0 & ,\ N=1,2\end{cases}$$

E'. $\Pr\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} | S_0 = S_B\right) = \left\langle \begin{matrix} 0 & , N \geq 2 \\ 1-q, & N = 1 \end{matrix} \right.$

F'. $\Pr\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} | S_0 = S_B\right) = \left\langle \begin{matrix} (1-q)\, p^{N-1}, & N \geq 2 \\ 0 & , N = 1 \end{matrix} \right.$

G'. $\Pr\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} | S_0 = S_B\right) = \left\langle \begin{matrix} (1-q)^{\left[\frac{N+1}{2}\right]} (1-p)^{\left[\frac{N}{2}\right]}, & N \geq 2 \\ 0 & , N = 1 \end{matrix} \right.$

H'.

$$\Pr\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} | S_0 = S_B\right) = \begin{cases} (1-q) \sum_{n=N-\left[\frac{N-2}{2}\right]}^{N-1} \frac{(n-1)!}{(2n-N-1)!(N-n)!} p^{2n-N-1} \left[(1-p)(1-q)\right]^{N-n}, & N \geq 4 \\ p(1-q)(1-p) & , N = 3 \\ 0 & , N = 1, 2 \end{cases}$$

I'.

$$\Pr\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} | S_0 = S_B\right) = \begin{cases} 1 - q^N - q^{N-1}(1-q) - q^{N-2}(1-q)\,p - (1-q)\,p^{N-1} - (1-q)^{\left[\frac{N+1}{2}\right]}(1-p)^{\left[\frac{N}{2}\right]} \\ - \sum_{n=N-\left[\frac{N-1}{2}\right]+1}^{N} \frac{(n-1)!}{(2n-N-2)!(N-n+1)!} q^{2n-N-2} \left[(1-q)(1-p)\right]^{N-n+1} \\ -(1-q) \sum_{n=N-\left[\frac{N-2}{2}\right]}^{N-1} \frac{(n-1)!}{(2n-N-1)!(N-n)!} p^{2n-N-1} \left[(1-p)(1-q)\right]^{N-n}, & N \geq 4 \\ 0 & , N = 1, 2, 3 \end{cases}$$

5 Conclusions

In this paper, we have discussed a simple heterogeneous agent mechanism with their applications. We introduce one more factor, length of evaluations on performances between strategies, which could interpret intensity of choice to switch strategies proposed by Brock and Hommes

(1997). Our results show the stickiness of states switching from one to another, and the longer length of evaluations on performances would generate more complex dynamic price fluctuations.

We then introduce different forms of transition matrix and inertia to investigate how the length of evaluations on performances would affect the transition among states. We connect our mechanism with Markov trajectory entropy proposed by Ekroot and Cover (1993) and provide total score and probability density functions of representations under two states as applications for the mechanism.

Further research would be to generalize probability density functions of representations under finite N states and provide more theoretical foundations for the mechanism.

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Appendix 1: Representations (Perron-Frobenius transition matrix)

Tie 1			Tie 2	
①	②	③	④	⑤
$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}1&1&0\\1&1&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}1&1&0\\0&1&0\\0&0&0\end{bmatrix}$
Tie 2				
⑥	⑦	⑧	⑨	⑩
$\begin{bmatrix}1&0&0\\1&1&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}1&0&1\\0&0&0\\1&0&1\end{bmatrix}$	$\begin{bmatrix}1&0&1\\0&0&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&0&0\\1&0&1\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&1&1\\0&1&1\end{bmatrix}$
Tie 2		Tie 3 [Group A]		
⑪	⑫	⑬	⑭	⑮
$\begin{bmatrix}0&0&0\\0&1&1\\0&0&1\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&1&0\\0&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0\\1&1&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0\\1&1&1\\0&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0\\1&1&1\\1&0&1\end{bmatrix}$
Tie 3 [Group A]				
⑯	⑰	⑱	⑲	⑳
$\begin{bmatrix}1&1&0\\1&1&1\\0&0&1\end{bmatrix}$	$\begin{bmatrix}1&1&0\\0&1&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0\\0&1&1\\0&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}$	$\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$
Tie 3 [Group B]				
㉑	㉒	㉓	㉔	㉕
$\begin{bmatrix}1&0&1\\1&1&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&0&1\\1&1&1\\0&1&1\end{bmatrix}$	$\begin{bmatrix}1&0&1\\1&1&0\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}$	$\begin{bmatrix}1&0&1\\0&1&1\\1&1&1\end{bmatrix}$
Tie 3 [Group B]			Tie 3 [Group C]	
㉖	㉗	㉘	㉙	㉚
$\begin{bmatrix}1&0&1\\0&1&1\\0&1&1\end{bmatrix}$	$\begin{bmatrix}1&0&1\\0&1&0\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&0&1\\0&1&0\\0&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&1\\1&0&1\end{bmatrix}$

Tie 3 [Group C]				
㉛	㉜	㉝	㉞	㉟
$\begin{bmatrix}1&1&1\\1&1&1\\0&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&1\\0&0&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&0\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\1&1&0\\0&1&1\end{bmatrix}$
㊱	㊲	㊳	㊴	㊵
$\begin{bmatrix}1&1&1\\1&1&0\\0&0&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\0&1&1\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\0&1&1\\1&0&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\0&1&1\\0&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$
㊶	㊷	㊸	㊹	
$\begin{bmatrix}1&1&1\\0&1&0\\1&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\0&1&0\\1&0&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\0&1&0\\0&1&1\end{bmatrix}$	$\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}$	

Parameter	Value
Fundamentalists' eagerness towards profits (α^f)	1
Positive-feedback traders' eagerness towards profits (α^c)	1
Wealth effect (β)	0.3
Funding rate (r^f, r^c)	0.1
Adjustment scale to expected price (ν, μ)	0.5
Intensity to switch strategy (φ)	0.1
Noise traders' reaction strength (γ)	3
Noise traders' proportion in the market (ξ)	0.3
Market's sensitivity to the demand of stock (θ)	0.001
Total number of traders in the market (n)	1000
Initial positive-feedback traders' proportion in the market (κ_0)	0.35
Initial fundamental value (p_0^*)	100
Initial stock price (p_0)	100

Table 1: Initial settings

	$l = 10$	$l = 500$
Augmented Dickey-Fuller test		
p-value	0.001	0.001
test statistics	-157.894	-118.121
critical value for the test	-2.861	-2.861
Phillips-Perron test		
p-value	0.001	0.001
test statistics	-157.895	-118.196
critical value for the test	-3.412	-3.412
Kwiatkowski-Phillips-Schmidt-Shin test		
p-value	0.100	0.010
test statistics	0.026	0.399
critical value for the test	0.146	0.146

Table 2: Unit root test results

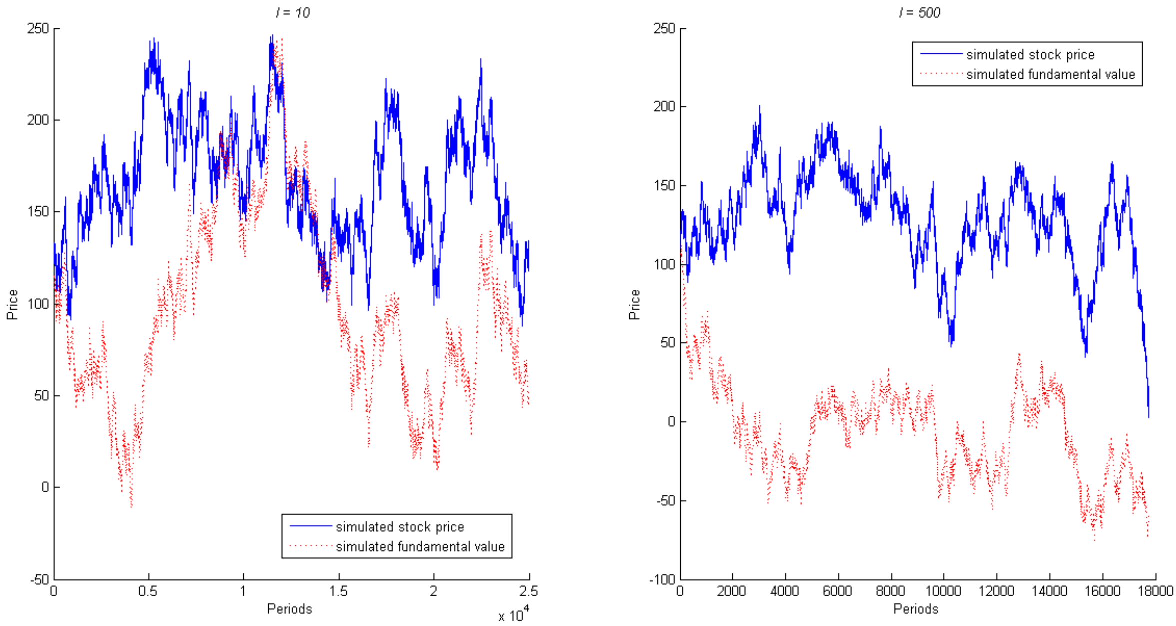

Figure 1：Scenarios for $l = 10,\ 500$

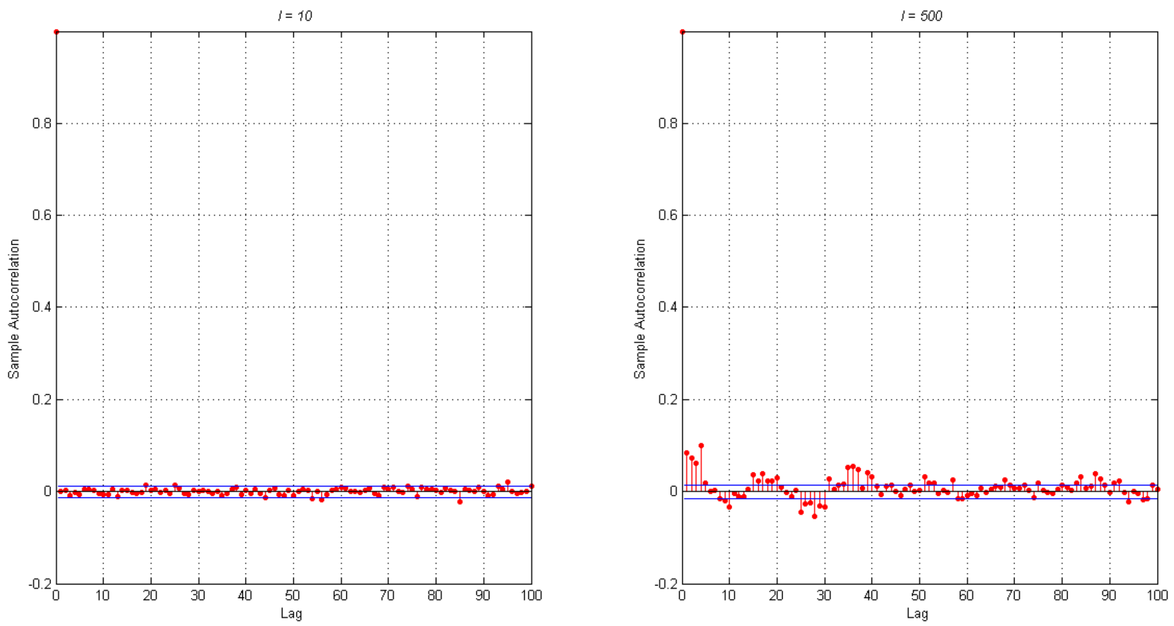

Figure 2：Autocorrelations of simulated returns for $l = 10,\ 500$

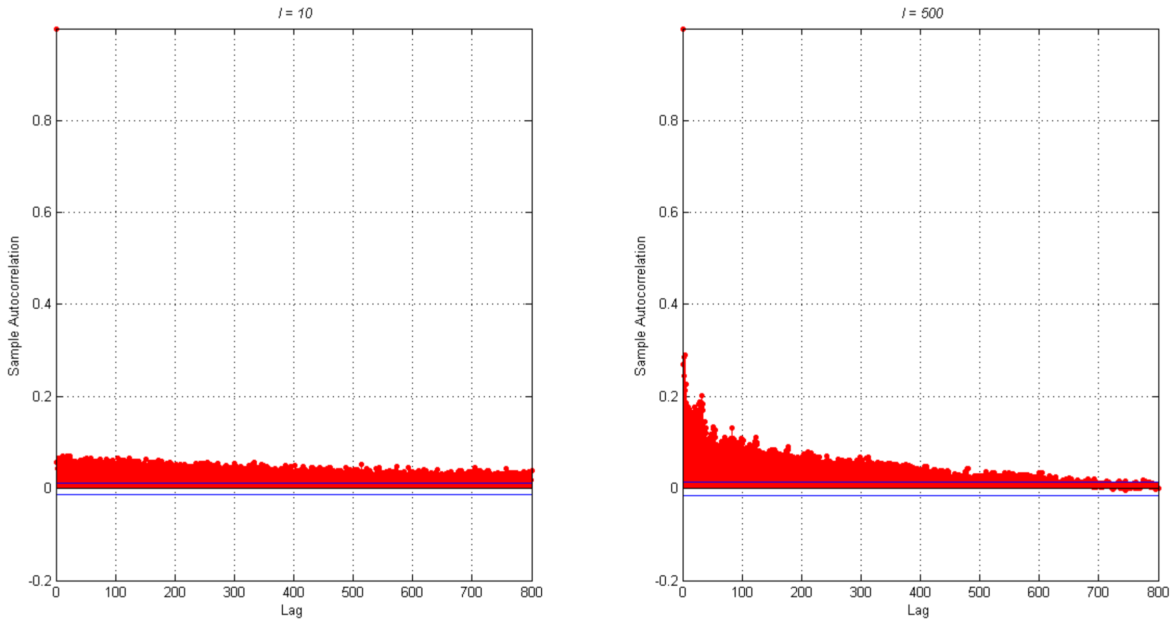

Figure 3：Autocorrelations of absolute simulated returns for $l = 10,\ 500$

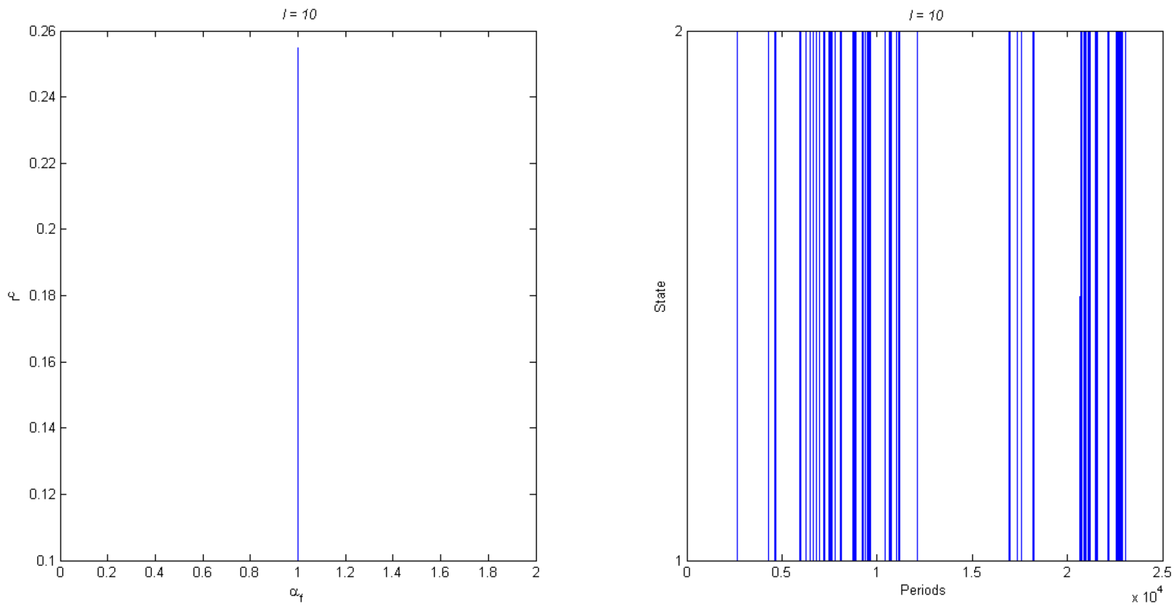

Figure 4：Trajectory of α_t^f and r_t^c (left) and transitions of states (right), $l = 10$

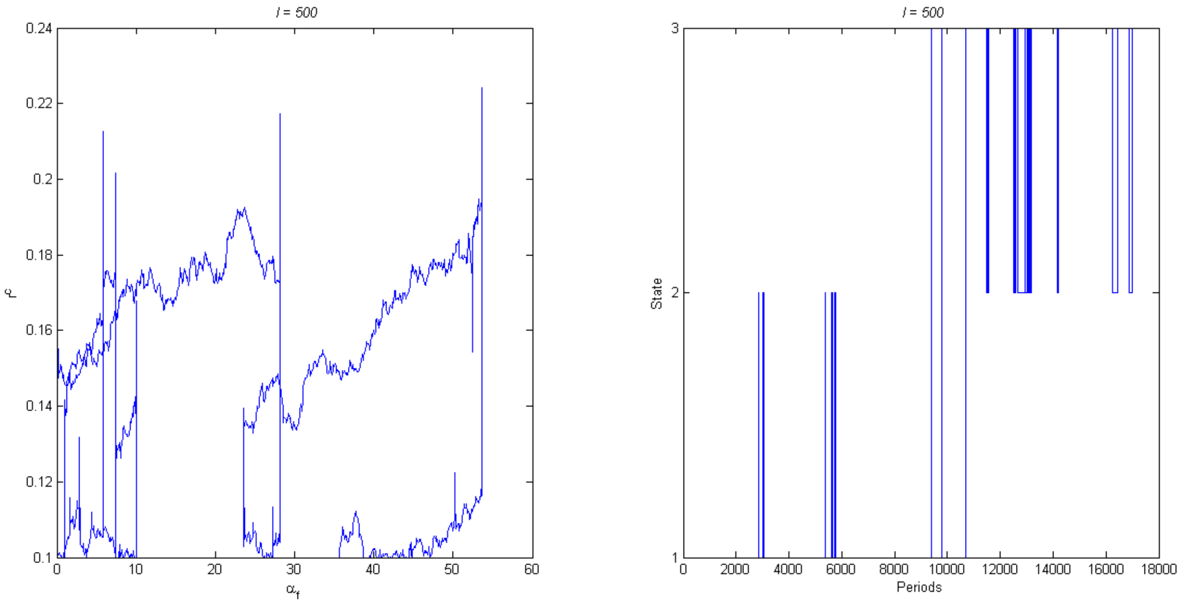

Figure 5：Trajectory of α_t^f and r_t^c (left) and transitions of states (right), $l = 500$

$$
\begin{array}{c}
\begin{array}{cc}
 & \begin{array}{ccc} S_p & S_C & S_F \end{array} \\
\begin{array}{c} S_p \\ S_C \\ S_F \end{array} & \begin{bmatrix} 0.9927 & 0.0073 & 0 \\ 0.0223 & 0.9777 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{array}
\\ l = 10
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cc}
 & \begin{array}{ccc} S_p & S_C & S_F \end{array} \\
\begin{array}{c} S_p \\ S_C \\ S_F \end{array} & \begin{bmatrix} 0.9986 & 0.0012 & 0.0002 \\ 0.0122 & 0.9523 & 0.0355 \\ 0.0001 & 0.0054 & 0.9945 \end{bmatrix}
\end{array}
\\ l = 500
\end{array}
$$

Figure 6：Markov transition matrix

$$
\begin{array}{c}
\begin{array}{cc}
 & \begin{array}{ccc} S_p & S_C & S_F \end{array} \\
\begin{array}{c} S_p \\ S_C \\ S_F \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{array}
\\ l = 10
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cc}
 & \begin{array}{ccc} S_p & S_C & S_F \end{array} \\
\begin{array}{c} S_p \\ S_C \\ S_F \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}
\end{array}
\\ l = 500
\end{array}
$$

Figure 7：Perron-Frobenius transition matrix

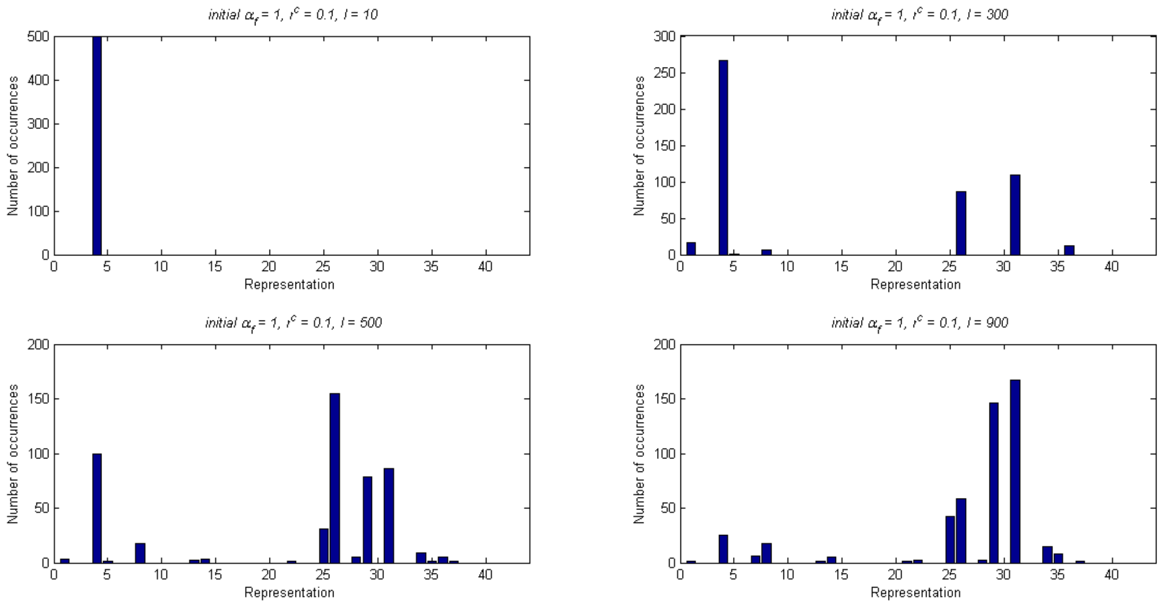

Figure 8 : State transitions as l vary, t-distributed

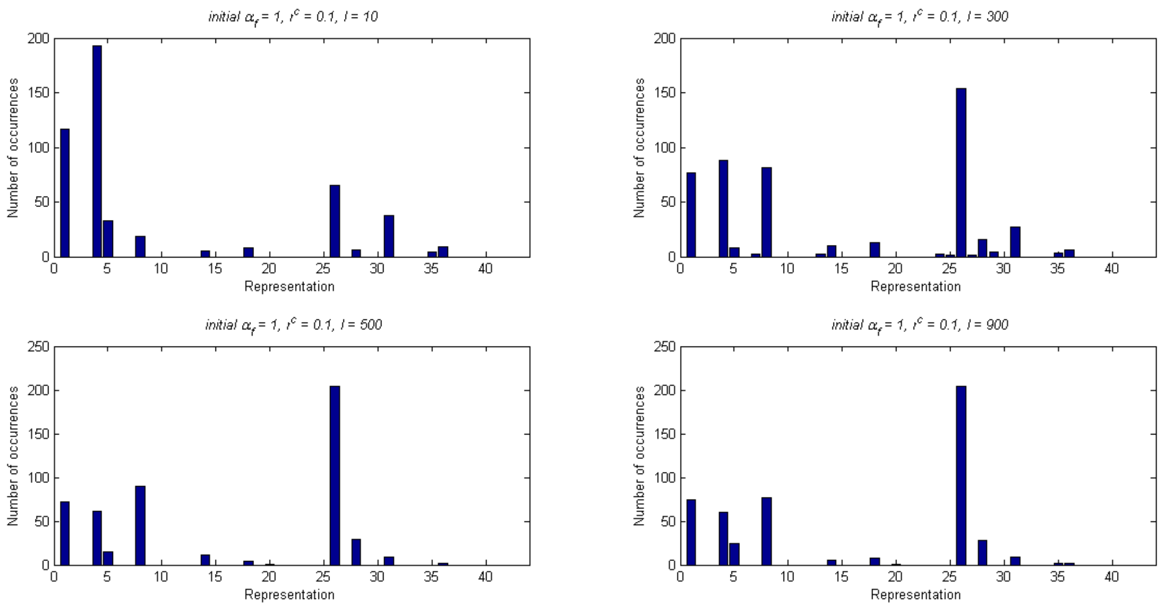

Figure 9 : State transitions as l vary, α-stable-distributed

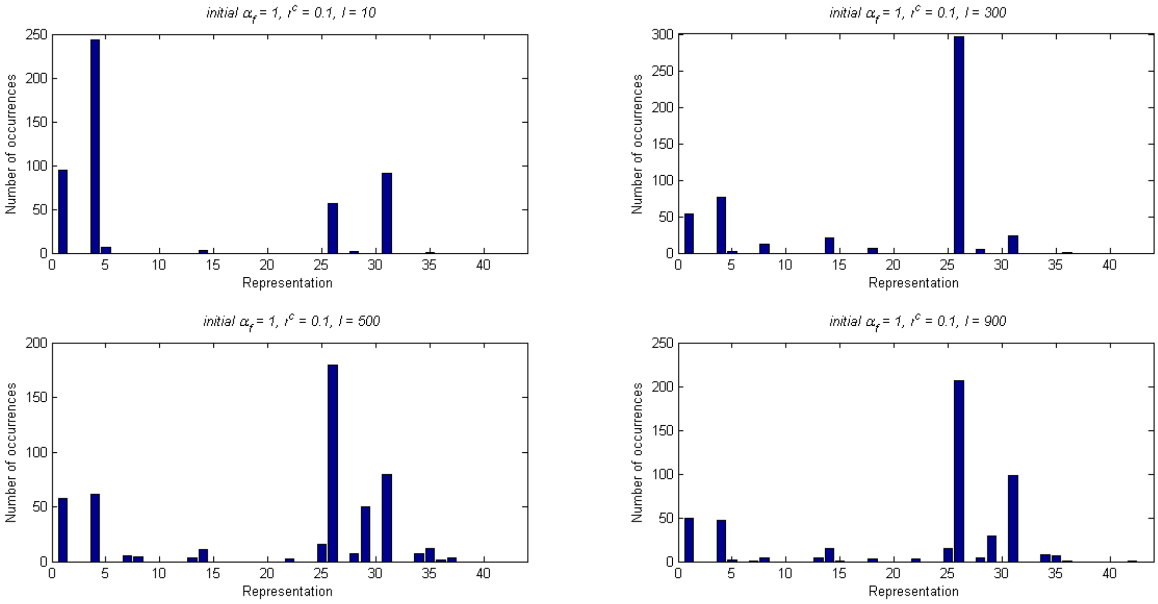

Figure 10：State transitions as l vary, GARCH-Normal process

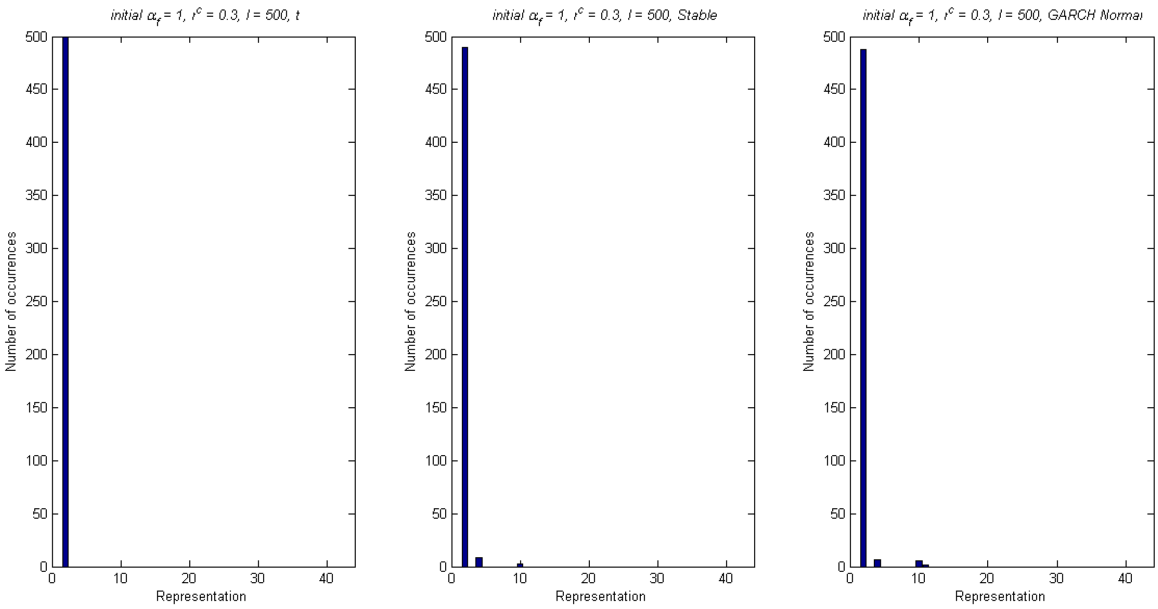

Figure 11：State transitions as initial r^c equals 0.3

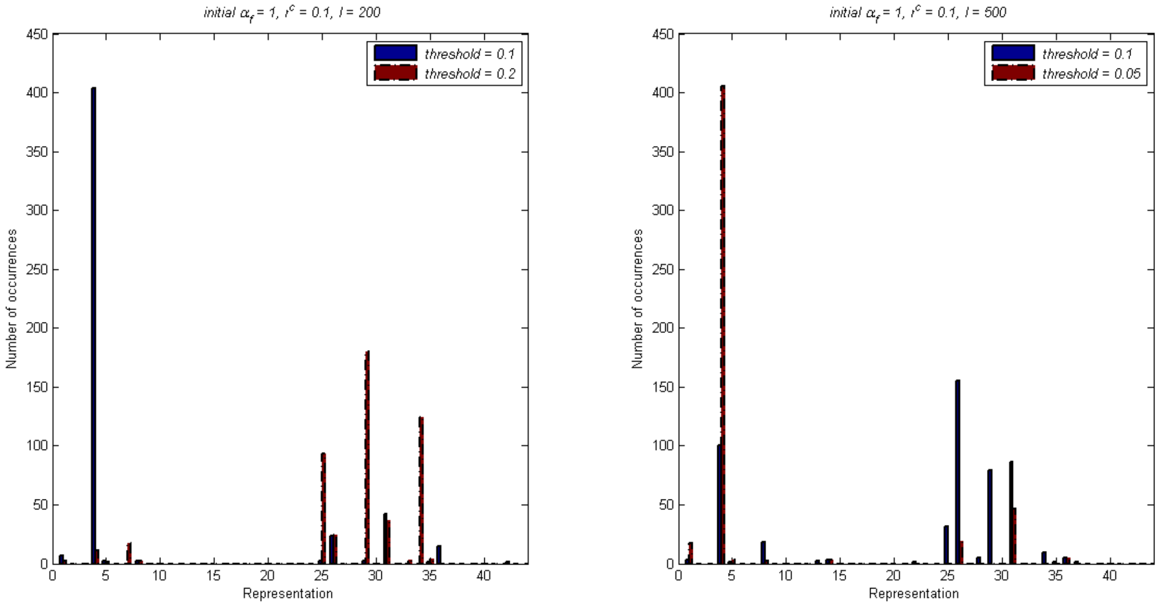

Figure 12：State transitions as ϑ vary, t-distributed

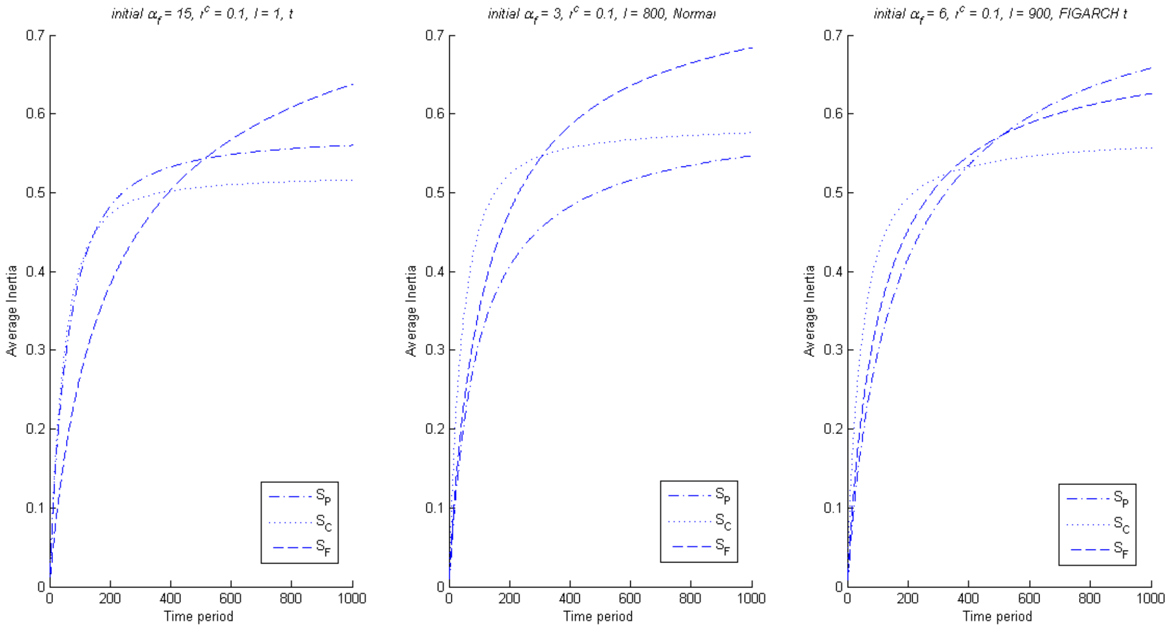

Figure 13：Inertia as initial r^c equals 0.1 (normal, t, and FIGARCH)

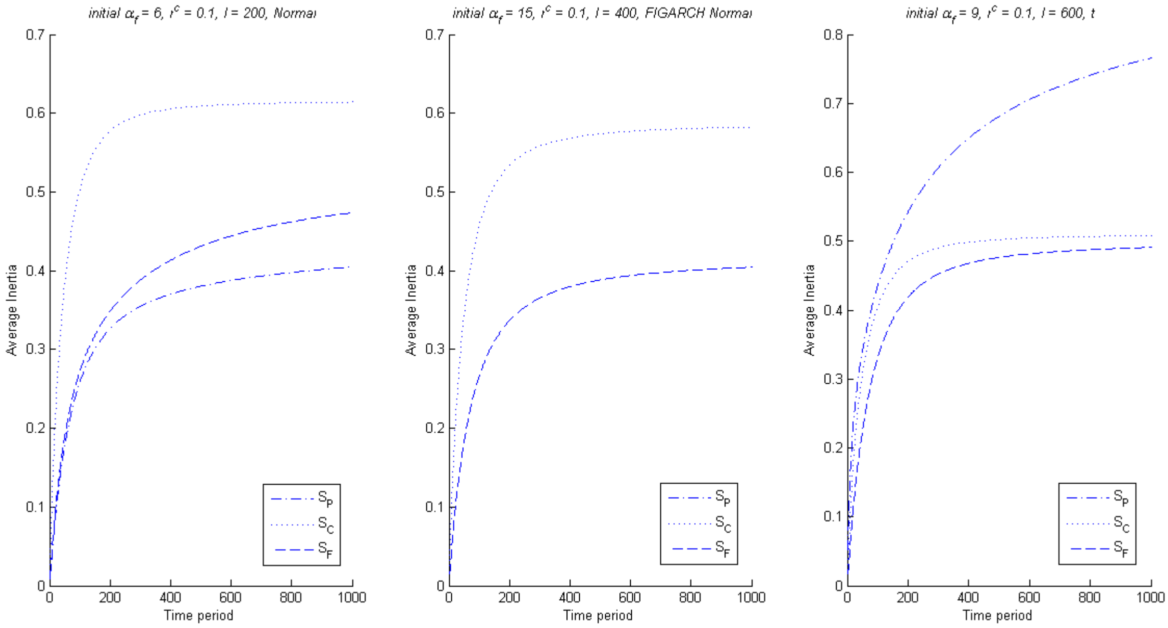

Figure 14：Inertia as initial r^c equals 0.1 (normal, *t*, and FIGARCH)

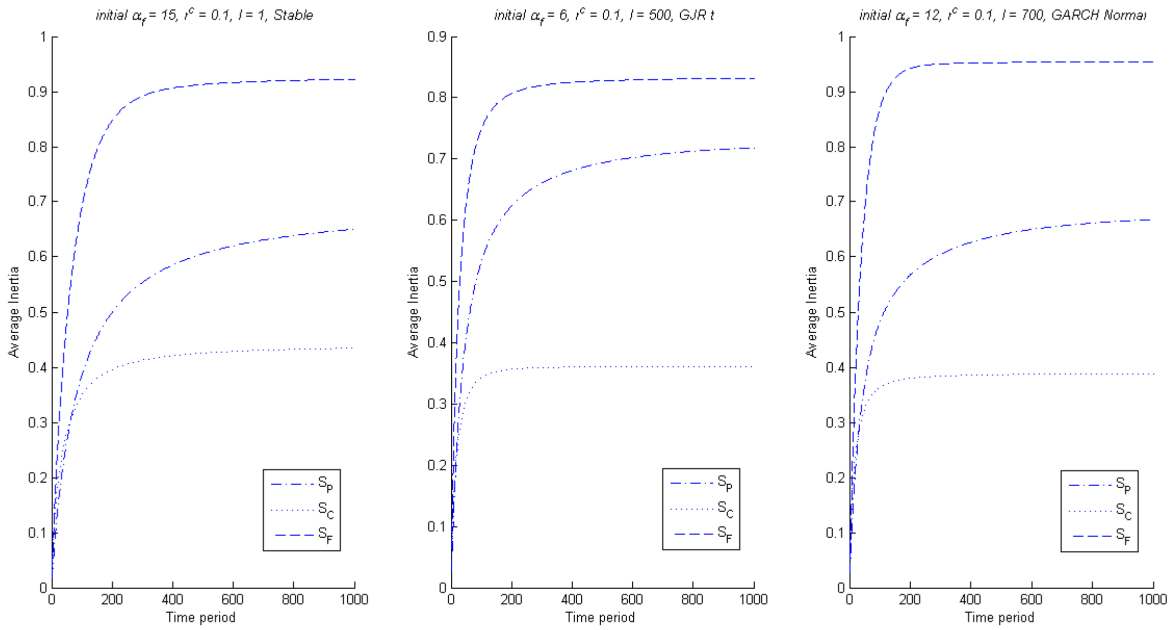

Figure 15：Inertia as initial r^c equals 0.1 (α -stable and GARCH)

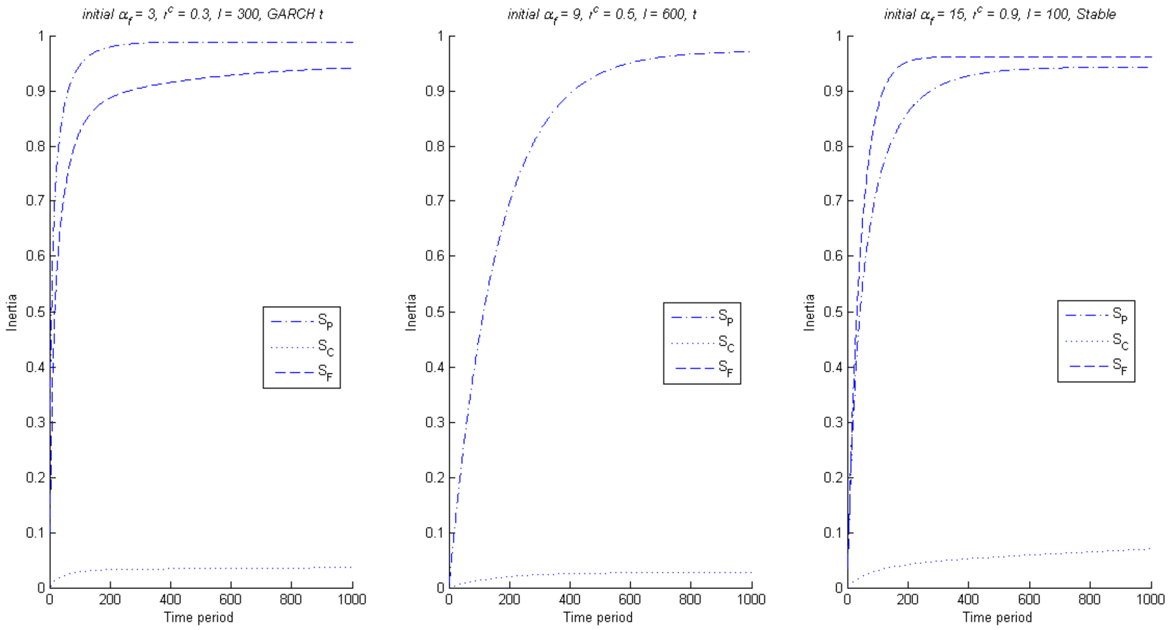

Figure 16：Inertia as initial r^c larger than 0.3

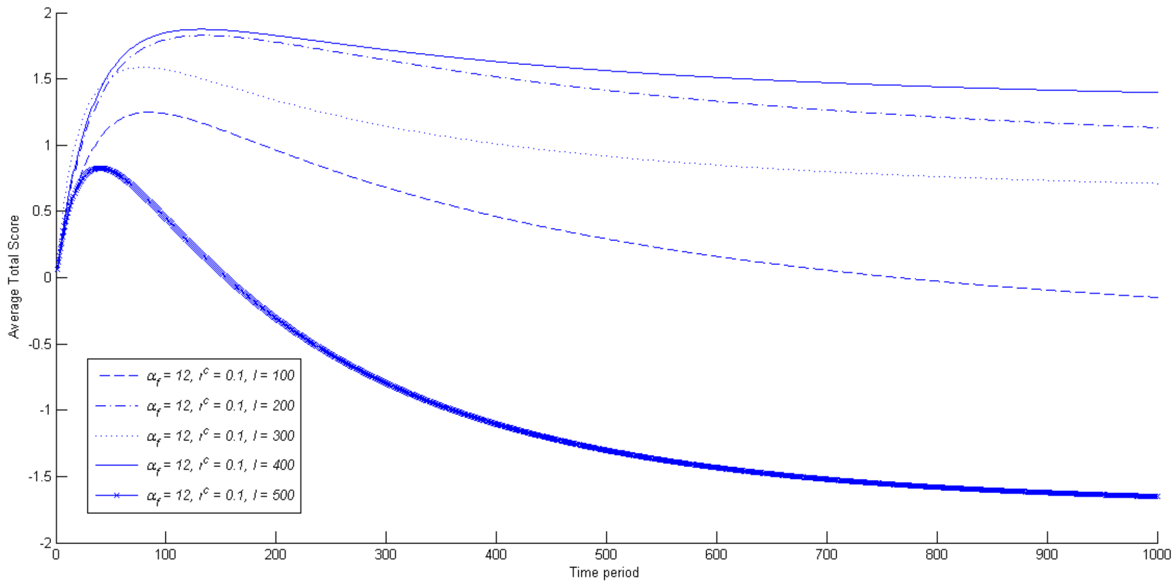

Figure 17：Total scores comparison, *t*–distributed

$$
\begin{array}{c c}
 & \begin{array}{cc} S_p & S_C \end{array} \\
\begin{array}{c} S_p \\ S_C \end{array} & \begin{bmatrix} 0.0779 & 5.9201 \\ 4.7942 & 0.2384 \end{bmatrix}
\end{array}
\qquad
\begin{array}{c c}
 & \begin{array}{ccc} S_p & S_C & S_F \end{array} \\
\begin{array}{c} S_p \\ S_C \\ S_F \end{array} & \begin{bmatrix} 0.0581 & 8.9424 & 15.2778 \\ 33.6855 & 0.5614 & 8.4871 \\ 39.0759 & 6.5746 & 0.0826 \end{bmatrix}
\end{array}
$$

$l = 10$ $\qquad$ $l = 500$

Figure 18：Markov trajectory entropy